Optimum power generation from a moving plasma

By JOSEPH L. NEURINGER
Plasma Propulsion Laboratory, Republic Aviation Corporation, Farmingdale, Long Island, New York

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The possibility exists of directly using the plasma, resulting from a controlled fusion reaction, to generate electricity by electromagnetic induction. Two special cases of a more general problem are considered here: (1) the extraction of optimum power from the steady one-dimensional flow of an incompressible inviscid plasma across a uniform transverse magnetic field in an externally loaded channel of arbitrarily varying cross-section, and (2) the extraction of optimum power from the steady one-dimensional flow of a compressible inviscid plasma across a uniform transverse magnetic field in a channel of uniform cross-section. In each case, the magnitude of the required external loading at optimum power operation is determined as a function of the parameters which characterize the hydromagnetic interaction. Also determined are the magnitudes of the terminal voltage, power, fluid mechanical to electrical conversion efficiency, and the variation of the fluid dynamical variables along the channel at optimum power.

1. Introduction

With controlled thermonuclear fusion a possibility for the future (Post 1956), the question may be raised as to the best, most efficient way of utilizing the energy liberated for electric power generation. The conventional steam-turbine method of generating electricity, which involves moving parts of heavy cumbersome machinery, may not in this case be the most practical and efficient way of electric generation. A possible method of electric generation which does not involve moving parts is generation by electromagnetic induction using the highly conducting plasma of the reaction products as the working fluid. The inductive action may be described as follows. A plasma moving perpendicular to a magnetic field, both flow and field being horizontal, has induced in it a vertical electric field. If the flow is in a channel with top and bottom walls conductors, and these walls are connected by an external load, then current will flow through the plasma and external load. Further, it is contemplated that by utilizing some of the mechanical energy of the exhausted ionized gases in a plasma pinch engine (Kunen 1958), electric power may be generated to be used either as a prime or auxiliary power unit for the engine system.

The dynamical state of the plasma, characterized by the state variables u_0, p_0, ρ_0 and T_0 exiting from a thermonuclear reactor or exhaust of a pinch engine will probably be fixed by the operating power level of the reactor. Also, the channel entrance width and length may be fixed by space and other limitations. We then pose the following question. For a channel of given entrance cross-section and length, what must be the shape of the channel, distribution of applied magnetic-field strength and magnitude of applied external loading so that maximum power may be extracted by the external load? The derivation of the conditions for maximum power transfer from conventional generators is developed in elementary texts, but for a plasma generator of the type considered here the situation is much more difficult to analyse because of the complex hydromagnetic interaction between a variable magnetic field and a compressible electrically conducting fluid continuum.

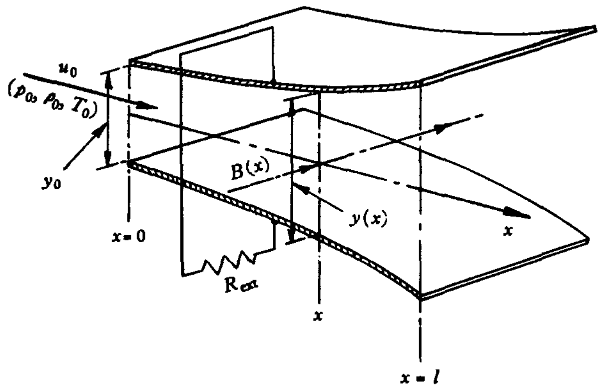

FIGURE 1. Diagram of plasma generator.

To simplify the analysis, consider the two-dimensional channel shown in figure 1. The channel is of unit breadth out of the paper. The plasma moves in the positive x-direction as shown, entering the channel of fixed opening y_0 with fixed values u_0, p_0, ρ_0 and T_0 of the state variables. A magnetic field, normal to the paper and directed into it, is assumed to be an unknown function of x only. The following simplifying assumptions are made:

(*a*) The flow is one-dimensional, i.e. the fluid dynamical state variables vary in the x-direction but not over the cross-section.

(*b*) The magnetic Reynolds number (see § 5) is small; that is, any effects on the fluid flow of secondary magnetic fields resulting from the induced current distribution are negligible, because either the secondary magnetic fields are too small or are in the wrong direction to produce appreciable effects.

(*c*) The induced currents in the x-direction are small compared to those induced in the y-direction, and are neglected.

(*d*) The plasma behaves like a perfect gas.

(*e*) The intrinsic properties σ and γ of the plasma are taken to be constant.

(*f*) The fluid is assumed inviscid and non-heat conducting, so that the only dissipation is electrical.

(*g*) The plasma is assumed to be electrically neutral so that no space-charge sheaths are developed near the conducting walls.

(*h*) The motion of the plasma through the channel is smooth, i.e. shock free.

(*i*) The channel walls are perfectly conducting.

The extremum problem, as outlined above, was formulated in Neuringer (1958), using the techniques of the calculus of variations to obtain the appropriate system of differential equations and boundary conditions. These equations are each of first order and highly non-linear, and they require machine calculation for their solution. In the special cases where the differential constraints, i.e. the fluid flow equations, are integrable, analytical solutions may be obtained, however, without resorting to the mathematical apparatus of the calculus of variations. The two special cases considered in this paper are:

(1) The flow of an incompressible, inviscid plasma across a uniform transverse magnetic field in a channel of fixed length in which the variation in cross-section along its length is arbitrarily prescribed.

(2) The flow of a compressible, inviscid plasma across a uniform transverse field in a channel of fixed length and uniform cross-section.

In each of these special cases, we shall determine the magnitude of the required external resistance, at optimum power generation, as a function of the parameters which characterize the hydromagnetic interaction. Also, we shall determine the resulting terminal voltage, power, fluid mechanical to electrical conversion efficiency, and distribution of the fluid dynamical state variables along the channel at optimum power.

2. Notation

The rational M.K.S. system is used throughout.

u = velocity
p = pressure
T = temperature
ρ = density
E = electric intensity
B = magnetic induction
H = magnetic intensity
J = current density
I = total current
r = resistance per unit length
R = total resistance
σ = plasma conductivity
μ_e = magnetic permeability
x = distance along channel
y = channel width
l = channel length
γ = ratio of specific heat
$\hat{P}$ = electric power
k = voltage appearing at the terminals of the generator
η = efficiency
m = constant mass flow rate through channel
c_a = speed of Alfvén waves
P = dimensionless pressure
U = dimensionless velocity
X = dimensionless distance along channel
k' = dimensionless terminal voltage
M_0 = Mach number of flow at channel entrance
$a = l/y_0$
δ = magnetohydrodynamic interaction parameter
α = parameter (function of γ and M_0)
β = parameter (function of γ and M_0)
0 = subscript representing entrance values

3. The circuit and magnetohydrodynamic equations

Let $r(x)$ represent the external unit resistance at station x along the channel. It may be defined as that element of the parallel resolution of the total external resistance which would carry the full current density associated with that station. Let $J(x)$ represent the induced current in the y direction per unit length of channel at station x. The external voltage drop is then $J(x)\,r(x)$. The voltage drop due to the internal resistance of the plasma is $J(x)\,y(x)/\sigma$. Kirchoff's law then gives for the closed circuit at station x:

$$Ey = Jr + \frac{Jy}{\sigma}, \tag{1}$$

where

$$E = uB. \tag{2}$$

Assumptions *i* and *c* imply

$$Jr = k, \tag{3}$$

where k, the voltage drop across the external load, is a constant to be determined.

Using (2) and (3) in (1), we obtain

$$J = \frac{\sigma}{y}\{uBy - k\}. \tag{4}$$

The power generated in the external circuit per unit length is

$$\frac{\hat{P}}{\text{length}} = rJ^2 = kJ = k\sigma\left\{uB - \frac{k}{y}\right\},$$

and the total power generated is

$$P = k\sigma \int_0^l \left(uB - \frac{k}{y}\right) dx. \tag{5}$$

The problem then is to find the velocity distribution $u(x)$ and the constant k which when inserted into (5) maximizes $\hat{P}$.

The one-dimensional compressible fluid flow equations, modified to account for the hydromagnetic interaction, are (Resler & Sears 1958):

Continuity equation

$$\rho u y = \rho_0 u_0 y_0 = m, \tag{6}$$

where m represents the constant mass flow rate through the channel.

Momentum equation

$$\rho u \frac{du}{dx} + \frac{dp}{dx} = -JB = -\left(\sigma u B^2 - \frac{\sigma k B}{y}\right), \tag{7}$$

where JB (with proper sign) represents the Lorentz force per unit volume exerted by the magnetic field on the fluid. The last term is obtained using (4) for J.

Energy equation

$$\frac{d}{dx}\left\{\left[\frac{p}{\gamma-1} + \tfrac{1}{2}\rho u^2\right] uy\right\} + \frac{d}{dx}\{puy\} + uJBy - \frac{J^2}{\sigma}\,y = 0, \tag{8'}$$

where the first term on the left-hand side represents the net flux of internal plus kinetic energy through the faces of a volume element, the second term represents the rate of mechanical work done by the pressure forces, the third term represents the rate of doing work by the Lorentz force, and the last term represents the Joulean dissipation. (8′) can be written, by means of (4), as

$$\frac{d}{dx}\left\{\left[\frac{\gamma}{\gamma-1}\frac{p}{\rho} + \tfrac{1}{2}u^2\right]\rho u y\right\} + k\sigma\left\{uB - \frac{k}{y}\right\} = 0. \tag{8}$$

An interesting observation may be made using the energy equation in the form (8). Substituting for the integrand in (5) using (8) and performing the integration, we obtain, using (6),

$$\hat{P} = \left\{\left[\frac{\gamma}{\gamma-1}\frac{p}{\rho}+\tfrac{1}{2}u^2\right]m\right\}_0 - \left\{\left[\frac{\gamma}{\gamma-1}\frac{p}{\rho}+\tfrac{1}{2}u^2\right]m\right\}_l. \tag{9}$$

That is, the power delivered to the external circuit is equal to the difference in the total, or stagnation, enthalpy flux at the entrance and exit of the channel.

4. Case 1: incompressible inviscid plasma flowing across a uniform transverse magnetic field in a channel of arbitrarily prescribed cross-section

Let $y(x)$ represent the arbitrarily prescribed variation of channel cross-section with length. The fluid density is uniform along the channel, so that the fluid velocity, from (6), is given by $u(x) = m/\rho_0 y$. Substituting for the velocity in the expression for the power (5), we obtain

$$\hat{P} = \left(\frac{k\sigma Bm}{\rho_0} - k^2\sigma\right)\int_0^l \frac{dx}{y(x)}. \tag{10}$$

For maximum power, we require $d\hat{P}/dk = 0$. Carrying out the differentation, we obtain

$$k = \frac{Bm}{2\rho_0} = \frac{u_0 y_0 B}{2}. \tag{11}$$

Two interesting observations may be made using (11).

(1) The value of the external or terminal voltage drop at maximum power is independent of the channel shape.

(2) Using (11) in conjunction with the condition that the induced e.m.f. at each station along the channel equals $uBy = u_0 B y_0 =$ constant, it is seen from Kirchoff's law that the internal voltage drop is equal to the external drop and hence a local matching theorem for maximum power is valid; i.e. the applied external unit loading, $r(x)$, at station x must be equal to the unit internal resistance of the working plasma at that station. It is not too surprising that in the case of incompressibility local matching is required for optimum power extraction. This is so because incompressibility yields the condition that the induced e.m.f. at every station along the channel is the same. This, coupled with the fact that the external voltage drop at each station must be identical (the channel walls are perfectly conducting), means that each station is uncoupled from the others and so can be treated as independently isolated. Conceptually, we may visualize the incompressible plasma generator as one consisting of a continuous distribution of independently acting elemental generators each with its own internal resistance. The elements (generator plus internal resistance) are then connected in parallel and the resulting combination feeding energy to an external load connected to the conducting walls.

The maximum power is, substituting for k from (11) into (10),

$$\hat{P} = \tfrac{1}{4}u_0^2 y_0^2 B^2 \sigma \int_0^l \frac{dx}{y(x)}. \tag{12}$$

The analysis leading to (12) was based purely on electrical energy considerations, i.e. Kirchoff's law and the definition of electrical power in terms of current and resistance. It is instructive, therefore, to reconsider the optimum power formula from a fluid mechanical energy viewpoint. Before doing so, the pressure distribution along the channel, at optimum power operation, is required. Substituting the value for k obtained in (11) into the momentum equation (7), we obtain the following differential equation for the pressure distribution along the channel:

$$\frac{dp}{dx} = \frac{m^2}{\rho_0 y^3}\frac{dy}{dx} - \frac{1}{2}\frac{\sigma B^2 m}{\rho_0 y}.$$

Integrating, and using the boundary condition $p = p_0$ at $x = 0$, we obtain

$$p = p_0 - \frac{m^2}{2\rho_0}\left\{\frac{1}{y^2} - \frac{1}{y_0^2}\right\} - \frac{\sigma B^2 m}{2\rho_0}\int_0^x \frac{d\xi}{y(\xi)}. \tag{13}$$

Now the fluid mechanical energy density of an incompressible, inviscid fluid is given by the Bernoulli expression $(p + \frac{1}{2}\rho_0 u^2)$. The energy flux at any station x is then $m/\rho_0(p + \frac{1}{2}\rho_0 u^2)$. Energy conservation requires that the difference in the fluid mechanical energy flux at the entrance and exit of the channel goes into electrical energy. Since we have already demonstrated that at optimum power operation the matching theorem is valid, the electrical energy must consist of two equal parts: (1) the electrical energy extracted by the external load, and (2) the electrical energy delivered to the fluid and which is dissipated internally in the form of heat. Mathematically, we have

$$\hat{P} = \frac{m}{2\rho_0}\{[p + \tfrac{1}{2}\rho_0 u^2]_0 - [p + \tfrac{1}{2}\rho_0 u^2]_l\}. \tag{14}$$

Substituting the expression (13) for p into (14), we find that $\hat{P}$ reduces identically to (12).

Incidentally, since the fluid pressure can not be less than zero at the channel exit, we have

$$p_0 - \frac{m^2}{2\rho_0}\left\{\frac{1}{y^2(l)} - \frac{1}{y_0^2}\right\} - \frac{\sigma B^2 m}{2\rho_0}\int_0^l \frac{d\xi}{y(\xi)} \geqslant 0 \tag{15}$$

as the condition which must be satisfied between the fluid mechanical, electromagnetic, and geometrical properties of the system.

Finally, as an interesting check on the inner consistency of what has gone before, we shall derive the maximum power formula using the techniques of lumped circuit theory. The external conductance in length dx is $dx/r(x)$. The total conductance is then

$$\int_0^l \frac{dx}{r(x)},$$

and the total external resistance is

$$R_{\text{ext}} = \left[\int_0^l \frac{dx}{r(x)}\right]^{-1}.$$

The current flow in the external load is

$$I = \int_0^l J(x)\,dx,$$

and since $J = k/r(x)$, we have $\quad I = k\int_0^l \frac{dx}{r(x)}.$

The total power delivered to the outside circuit is I^2R_{ext}. Using the above expressions for I and R_{ext}, we obtain

$$\hat{P} = I^2R_{\text{ext}} = k^2\int_0^l \frac{dx}{r(x)}. \tag{16}$$

Now at optimum power, $k = \frac{1}{2}(u_0y_0B)$ and $r(x) = y(x)/\sigma$ (local matching). Substituting for k and $r(x)$ into (16), we see that (16) reduces to (12).

For later comparison with the compressible flow case, we repeat the principal result of this section: for an incompressible plasma, the condition for optimum power requires that the external load be equal to the internal resistance of the plasma.

5. Case 2: compressible inviscid plasma moving across uniform transverse magnetic field in a channel of uniform cross-section

Analysis

We treat the optimization problem for the compressible flow case in a different manner than the incompressible case. Instead of seeking to maximize the power integral (5), we shall avoid a complicated integration by considering the maximization of the equivalent expression for the power, namely (9). (9) states that the electric power delivered to the external load is a maximum when the total enthalpy flux at the channel exit is a minimum. However, before we can minimize the total exit enthalpy flux, we must first integrate the fluid mechanical flow equations (6), (7) and (8), in order to determine the appropriate expressions (to be inserted into (9)), for the velocity, pressure and density of the plasma at the channel exit.

Before integrating, it is convenient to introduce the following dimensionless variables:

$$U = \frac{u}{u_0}; \quad X = \frac{x}{y_0}; \quad P = \frac{p}{\rho_0 u_0^2};$$

and the following dimensionless parameters:

$$k' = \frac{u_0y_0B}{k}; \quad \frac{l}{y_0} = a; \quad \frac{p_0}{\rho_0u_0^2} = \frac{1}{\gamma M_0^2}; \quad \frac{B^2y_0^2\sigma}{m}\frac{l}{y_0} = \delta.$$

The meaning of the first three parameters is fairly obvious and needs no further consideration. Let us briefly consider the significance of the fourth. Remembering that $B = \mu_e H$ and the definition of m, we can write δ as

$$\delta = \frac{B^2y_0^2\sigma}{m}\frac{l}{y_0} = \frac{\sigma\mu_e^2H^2l}{\rho_0u_0} = (\mu_e\sigma l u_0)\left(\frac{\mu_eH^2}{\rho_0u_0^2}\right).$$

In the next to the last form, δ represents the ratio of the electrical body force to the dynamic or inertial force on the conducting fluid (Resler & Sears 1958). Hence, when δ is large we can expect the applied magnetic field to produce significant effects on the fluid motion.

Consider now the resolved form of δ. The term $(\mu_e \sigma l u_0)$ is defined as the magnetic Reynolds number (based on channel length). It is a measure of the effect of the hydromagnetic interaction on the magnetic field, and may be interpreted as the ratio of the motion induced magnetic field strength to the applied magnetic field strength. The term $(\mu_e H^2/\rho_0 u_0^2)$ can be written as

$$\frac{\mu_e H^2}{\rho_0 u_0^2} = \frac{c_a^2}{u_0^2},$$

where $c_a = \sqrt{(\mu_e H^2/\rho_0)}$ represents the speed of propagation of certain magnetohydrodynamic waves, called Alfvén waves. Thus $\sqrt{(\rho_0 u_0^2/\mu_e H^2)}$ is analogous to a Mach number where the usual speed of sonic disturbances is replaced by the speed of the Alfvén wavelike disturbances. The dimensionless number, δ, we shall henceforth call the magnetohydrodynamic interaction parameter.

In terms of the dimensionless variables and parameters, equations (7) and (8) become

$$\frac{dU}{dX} + \frac{dP}{dX} + \frac{\delta}{a}\left(U - \frac{1}{k'}\right) = 0, \tag{7'}$$

$$\frac{d}{dX}\left[\frac{\gamma}{\gamma-1}PU + \tfrac{1}{2}U^2\right] + \frac{\delta}{ak'}\left(U - \frac{1}{k'}\right) = 0. \tag{8'}$$

Eliminating $(U - 1/k')$ from (7′) using (8′), (7′) becomes

$$\frac{dU}{dX} + \frac{dP}{dX} - k'\frac{d}{dX}\left[\frac{\gamma}{\gamma-1}PU + \tfrac{1}{2}U^2\right] = 0.$$

Integrating, using the boundary conditions $U = 1$ and $P = 1/\gamma M_0^2$ at $X = 0$, we obtain

$$P = \frac{(\alpha + k'\beta) - (U - (\tfrac{1}{2}k')\,U^2)}{1 - [\gamma/(\gamma-1)]\,k'U}, \tag{17}$$

where

$$\alpha = 1 + \frac{1}{\gamma M_0^2}; \quad \beta = -\left(\frac{1}{\gamma-1}\frac{1}{M_0^2} + \frac{1}{2}\right).$$

Substituting for P in (7′) using (17), we obtain, after some differentiations and combination,

$$\frac{dU}{dX}\left[\left(1 - \frac{\gamma}{\gamma-1}k'U\right)^2 - (1 - k'U)\left(1 - \frac{\gamma}{\gamma-1}k'U\right)\right.$$
$$\left. + \frac{\gamma}{\gamma-1}k'\left\{(\alpha + k'\beta) - \left(U - \frac{k'}{2}U^2\right)\right\}\right] + \frac{\delta}{a}\left(1 - \frac{\gamma}{\gamma-1}k'U\right)^2\left(U - \frac{1}{k'}\right) = 0. \tag{18}$$

Integrating (18), using the boundary condition $U = 1$ at $X = 0$, we obtain the following expression for U as an implicit function of X:

$$\left\{\frac{\gamma(\gamma-1)}{2} - 1 - \gamma(\gamma-1)\,k'(\alpha + k'\beta)\right\}\log\left(\frac{1 - k'U}{1 - k'}\right)$$
$$+ \left\{\frac{\gamma-1}{\gamma} - \frac{(\gamma^2+1)(\gamma-1)}{2\gamma} + \gamma(\gamma-1)\,k'(\alpha + k'\beta)\right\}\log\left(\frac{1 - [\gamma/(\gamma-1)]\,k'U}{1 - [\gamma/(\gamma-1)]\,k'}\right)$$
$$+ \left\{\frac{\gamma^2}{\gamma-1}k'^2(\alpha + k'\beta) - \frac{\gamma+1}{2}k'\right\}\left\{\frac{U-1}{(1 - [\gamma/(\gamma-1)]\,k'U)(1 - [\gamma/(\gamma-1)]\,k')}\right\} = \frac{\delta}{a}X. \tag{19}$$

The total exit enthalpy flux, in terms of the dimensionless variables, is

$$mu_0^2\left\{\frac{\gamma}{\gamma-1}P(a)\,U(a)+\tfrac{1}{2}U^2(a)\right\},$$

where the argument a is the value of X at the channel exit. The necessary condition that the total exit enthalpy flux be a minimum requires the derivative with respect to k' of the above expression to equal zero, or

$$\frac{\gamma}{(\gamma-1)}\left\{\frac{dP(a)}{dk'}U(a)+P(a)\frac{dU(a)}{dk'}\right\}+U(a)\frac{dU(a)}{dk'}=0. \tag{20}$$

Differentiate (17) with respect to k' to obtain $dP(a)/dk'$. Substituting for $dP(a)/dk'$ into (20), rearranging and simplifying, we obtain,

$$\mathbf{A}\frac{dU(a)}{dk'}+\mathbf{B}=0, \tag{20$'$}$$

where

$$\mathbf{A}=\frac{2\gamma(\gamma-1)(\alpha+k'\beta)-2(\gamma^2-1)\,U+\gamma(\gamma+1)\,k'U^2}{2(\gamma-1)^2\{1-[\gamma/(\gamma-1)]\,k'U\}^2},$$

$$\mathbf{B}=\frac{2\gamma(\gamma-1)\,\beta U+2\gamma^2\alpha U^2-\gamma(\gamma+1)\,U^3}{2(\gamma-1)^2\{1-[\gamma/(\gamma-1)]\,k'U\}^2}.$$

Differentiating (19) with respect to k', rearranging and simplifying, we obtain

$$\{\mathbf{1}+\mathbf{2}+\mathbf{3}\}\frac{dU(a)}{dk'}+\{\mathbf{4}+\mathbf{5}+\mathbf{6}+\mathbf{7}+\mathbf{8}+\mathbf{9}\}=0, \tag{22}$$

where

$$\mathbf{1}=\frac{-k'}{(1-k'U)}\left\{\frac{\gamma(\gamma-1)}{2}-1-\gamma(\gamma-1)\,k'(\alpha+k'\beta)\right\};$$

$$\mathbf{2}=\frac{-\gamma k'}{(1-[\gamma/(\gamma-1)]\,k'U)}\left\{\frac{1}{\gamma}-\frac{\gamma^2+1}{2\gamma}+\gamma k'(\alpha+k'\beta)\right\};$$

$$\mathbf{3}=\frac{\{2\gamma^2k'^2(\alpha+k'\beta)-(\gamma^2-1)\,k'\}}{2(\gamma-1)(1-[\gamma/(\gamma-1)]\,k'U)^2};$$

$$\mathbf{4}=-\gamma(\gamma-1)(\alpha+2k'\beta)\log\left(\frac{1-k'U}{1-k'}\right);$$

$$\mathbf{5}=\gamma(\gamma-1)(\alpha+2k'\beta)\log\left(\frac{1-[\gamma/(\gamma-1)]\,k'U}{1-[\gamma/(\gamma-1)]\,k'}\right);$$

$$\mathbf{6}=\left\{\frac{\gamma(\gamma-1)}{2}-1-\gamma(\gamma-1)\,k'(\alpha+k'\beta)\right\}\frac{(1-U)}{(1-k'U)(1-k')};$$

$$\mathbf{7}=\frac{\{1-\frac{1}{2}(\gamma^2+1)+\gamma^2k'(\alpha+k'\beta)\}\{1-U\}}{(1-[\gamma/(\gamma-1)]\,k'U)(1-[\gamma/(\gamma-1)]\,k')};$$

$$\mathbf{8}=\frac{\{\gamma^2/(\gamma-1)(2\alpha k'+3k'^2\beta)-\frac{1}{2}(\gamma+1)\}(U-1)}{(1-[\gamma/(\gamma-1)]\,k'U)(1-[\gamma/(\gamma-1)]\,k')};$$

$$\mathbf{9}=\frac{[\gamma^2/(\gamma-1)\,k'^2(\alpha+k'\beta)-\frac{1}{2}(\gamma+1)\,k'][\gamma/(\gamma-1)(U-1)\{(1+U)-[2\gamma/(\gamma-1)]\,k'U\}]}{(1-[\gamma/(\gamma-1)]\,k'U)^2(1-[\gamma/(\gamma-1)]\,k')^2};$$

Eliminating $dU(a)/dk'$ from (20′) and (22), we obtain

$$\{\mathbf{1}+\mathbf{2}+\mathbf{3}\}\,\mathbf{B}-\{\mathbf{4}+\mathbf{5}+\mathbf{6}+\mathbf{7}+\mathbf{8}+\mathbf{9}\}\,\mathbf{A} = 0. \tag{23}$$

Equation (23), together with equation (19) (evaluated at $X = \alpha$), form a system of two simultaneous transcendental equations in the two unknowns k' and $U(a)$ as functions of the parameters γ, M_0, and δ.

Results and discussion

A specification of the 'input' parameters γ, M_0, δ and a, and the quantities k' and $U(a)$ obtained from the simultaneous solution of (23) and (19) is sufficient to determine completely all of the electrical and fluid mechanical properties of the interaction. In particular, we shall now derive, in terms of these parameters, explicit formulas for (1) the ratio of the required external resistance at optimum power to the internal resistance of the equivalent incompressible flow, (2) the power, and (3) the generator efficiency.

Eliminating J from (1) and (3), using (2), and solving for r, we obtain

$$r = \frac{ky_0}{\sigma(uBy_0-k)}.$$

Proceeding similarly as in § 4, we sum in parallel the continuous distribution of these elemental resistances, and obtain

$$R_{\text{ext}} = \frac{ky_0}{\sigma}\left[\int_0^l (uBy_0-k)\,dx\right]^{-1}.$$

An element of internal resistance is y_0/σ. The total resistance, R_{int}, of an incompressible plasma moving in a channel of uniform width y_0 is then $y_0/\sigma l$. Forming the resistance ratio, integrating the second term of the above integral, and transforming to dimensionless variables, we obtain

$$\frac{R_{\text{ext}}}{R_{\text{int}}} = \left[\frac{k'}{a}\int_0^a U\,dX-1\right]^{-1} = \left[\frac{k'}{a}\int_1^{U(a)} U\frac{dX}{dU}\,dU-1\right]^{-1}. \tag{24}$$

Forming dX/dU as an explicit function of U by differentiating (19), substituting into the latter integral and performing the integration, we finally obtain

$$\frac{R_{\text{int}}}{R_{\text{ext}}} = \frac{k'}{\delta}[\mathbf{10}+\mathbf{11}+\mathbf{12}]-1, \tag{25}$$

where $\mathbf{10} = \left\{\frac{\gamma(\gamma-1)}{2k'}-\frac{1}{k'}-\gamma(\gamma-1)(\alpha+k'\beta)\right\}\log\left(\frac{1-k'U}{1-k'}\right);$

$$\mathbf{11} = \left\{\gamma(\gamma-1)(\alpha+k'\beta)-\frac{(\gamma-1)^2(\gamma+1)}{2k'\gamma}\right\}\log\left(\frac{1-[\gamma/(\gamma-1)]\,k'U}{1-[\gamma/(\gamma-1)]\,k'}\right);$$

$$\mathbf{12} = (U-1)\left\{\frac{\gamma k'(\alpha+k'\beta)-(\gamma^2-1)/2\gamma}{(1-[\gamma/(\gamma-1)]\,k'U)\,(1-[\gamma/(\gamma-1)]\,k')}-\frac{(\gamma+1)}{2\gamma}\right\};$$

where it is understood that U is evaluated at $X = a$.

We emphasize once again that in the above formulas, R_{int} does not represent the internal resistance of the compressible plasma, but the resistance which would obtain had the plasma been treated as incompressible. It is clear that only in the case of incompressibility, where the induced e.m.f. at each station is the same, can we add in parallel the continuous distribution of elemental internal resistances. We chose to normalize R_{ext} with respect to the incompressible R_{int} because the latter is a resistance property of the system which is independent of the hydromagnetic interaction and depends only on the conductivity of the fluid and the geometry of the system (i.e. the width to length ratio).

The formula for the optimum power is obtained as follows:

$$\hat{P} = I^2 R_{\text{ext}} = k^2 \int_0^l \frac{dx}{r(x)} \quad (\text{see §4}).$$

Substituting the expression for $r(x)$, obtained above, into the integrand, and transforming to dimensionless variables, we obtain

$$\hat{P} = \frac{B^2 u_0^2 \sigma y_0 l}{k'^2} \left[\frac{k'}{a} \int_0^a U\, dX - 1 \right].$$

The bracketed term is precisely $(R_{\text{int}}/R_{\text{ext}})$; hence

$$\hat{P} = \frac{B^2 u_0^2 \sigma y_0 l}{k'^2} \frac{R_{\text{int}}}{R_{\text{ext}}}. \tag{26}$$

We define the efficiency or effectiveness, η, of the generator as

$$\eta = \frac{\text{Optimum electric power generated}}{\text{Total input enthalpy flux}}.$$

Forming the ratio of (26) to the total input enthalpy flux, i.e.

$$m \left[\frac{\gamma}{\gamma - 1} \frac{p_0}{\rho_0} + \tfrac{1}{2} u_0^2 \right],$$

and transforming to dimensionless variables, we obtain

$$\eta = -\frac{\delta}{\beta k'^2} \frac{R_{\text{int}}}{R_{\text{ext}}} = 1 + \left[\frac{(\gamma/\gamma - 1)\, P(a)\, U(a) + \tfrac{1}{2} U^2(a)}{\beta} \right], \tag{27}$$

where the last term represents the difference in entrance and exit stagnation enthalpy fluxes divided by the entrance flux. It is seen that the efficiency η is a complicated function (because of the complexity of the resistance ratio term) of the two dimensionless parameters δ and M_0, characterizing respectively the hydromagnetic and compressive properties of the system.

Solutions of the system of equations (23) and (19) (evaluated at $X = a$) for k' and $U(a)$ were obtained on an I.B.M. 704 computer. Solutions were obtained for the particular Mach number $M_0 = 0{\cdot}3$ and for a γ equal to $\frac{5}{3}$ (corresponding to the assumption that the plasma be treated as a monatomic perfect gas). δ was allowed to vary continuously until a maximum $\delta_{\max} \sim 30$ was reached beyond which no solutions could be generated.

Consider table 1 which lists the magnitudes of all the dimensionless electrical and fluid mechanical quantities. It is seen that the solution exhibits an asymptotic behaviour near $\delta_{crit} \sim 4$. In particular, if attention is focused on the column of exit Mach numbers $M(a)$, it is noted that for the second branch ($\delta > 4$), $M(a)$ is always equal to one. We conclude, therefore, the existence of a finite range of δ, i.e. $\delta_{crit} \leqslant \delta \leqslant \delta_{max}$, where the required external loading at optimum power is that which forces a Mach number one flow at the exit.

δ	k'	$2/k'$	R_{ext}/R_{int}	I/I_{inc}	$\eta\%$	$U(a)$	$P(a)$	$M(a)$
0·1	2·0017	0·99914	0·99166	1·0075	0·1466	1·0067	6·6096	0·30230
0·2	2·0035	0·99823	0·98321	1·0153	0·29519	1·0136	6·5515	0·30468
0·4	2·0076	0·99623	0·96585	1·0315	0·59858	1·0283	6·4321	0·30976
0·6	2·0122	0·99394	0·94781	1·0487	0·91076	1·0441	6·3080	0·31514
0·8	2·0175	0·99131	0·92901	1·0671	1·2324	1·0612	6·1786	0·32101
1·0	2·0236	0·98832	0·90942	1·0868	1·5642	1·0799	6·0434	0·32745
2·0	2·0736	0·96450	0·79390	1·2149	3·4129	1·2093	5·2425	0·37296
4·0	2·0077	0·99617	0·62218	1·6011	9·2909	2·7903	1·6742	1·0000
6·0	1·4202	1·4083	1·2769	1·1029	13·572	2·7237	1·6342	1·0000
8·0	1·2407	1·6120	1·9160	0·84134	15·801	2·6882	1·6130	1·0000
10·0	1·1562	1·7299	2·5444	0·67987	17·127	2·6671	1·6002	1·0000
12·0	1·1083	1·8045	3·1645	0·57024	17·983	2·6533	1·5920	1·0000
14·0	1·0783	1·8547	3·7780	0·49093	18·565	2·6438	1·5863	1·0000
16·0	1·0582	1·8899	4·3865	0·43085	18·976	2·6372	1·5823	1·0000
18·0	1·0442	1·9154	4·9893	0·38389	19·274	2·6323	1·5794	1·0000
20·0	1·0340	1·9342	5·5887	0·34608	19·496	2·6287	1·5772	1·0000
25·0	1·0186	1·9636	7·0735	0·27760	19·845	2·6230	1·5738	1·0000
30·0	1·0105	1·9791	8·5437	0·23165	20·358	2·6146	1·5687	1·0000

TABLE 1. List of the dimensionless electrical and fluid dynamical variables. $M_0 = 0{\cdot}3$; $\gamma = \frac{5}{3}$

At least for the branch characterized by $M(a) = 1$, the condition which determines δ_{max} may be formulated as follows. The Mach number at any point in the channel in terms of the dimensionless fluid dynamical variables P and U is given by $M = \sqrt{(U/\gamma P)}$. The condition $M(a) = 1$ at the exit yields $P(a) = U(a)/\gamma$. Substituting for $P(a)$ in (23), we obtain

$$\frac{U(a)}{\gamma} = \frac{(\alpha + k'\beta) - [U(a) - (\tfrac{1}{2}k')\,U^2(a)]}{1 - [\gamma/(\gamma-1)]\,k'U(a)}. \tag{28}$$

The simultaneous solution of (19), (23), and (28) yields δ_{max} and the corresponding limiting values of k' and $U(a)$.

Let us consider now the electrical quantities. Referring to the definitions of k and k', both compressible and incompressible, it is seen that $2/k'$ represents the ratio of the compressible terminal voltage to the terminal voltage obtained by assuming the gas to be incompressible. Similarly, I/I_{inc}, the ratio of the compressible to the incompressible current, is given by $(2/k')(R_{int}/R_{ext})$. It should be emphasized that incompressibility is defined and used here in the sense of § 4; i.e. the density of the fluid is everywhere constant in space and time and not as the limit flow approached as the Mach number is made to approach zero.

Plots of the resistance, voltage and current ratios as functions of the magnetohydrodynamic interaction parameter δ are shown in figure 2. Several general observations can be made. First, for weak interactions (i.e. δ small), the fluid behaves as if it were incompressible; that is, the resistance, terminal voltage and current remain close to the corresponding incompressible values. Secondly, the asymptotic behaviour is clearly marked and is represented in the resistance and current curves as forming cusps at the critical δ. Thirdly, for large δ, the behaviour of the solution is very much different from the corresponding incompressible case. The terminal voltage ratio approaches the asymptotic value two while the external load ratio continues to increase sharply with δ. This behaviour is very significant, for it indicates (as is often done in fluid dynamics in order to obtain a first estimate

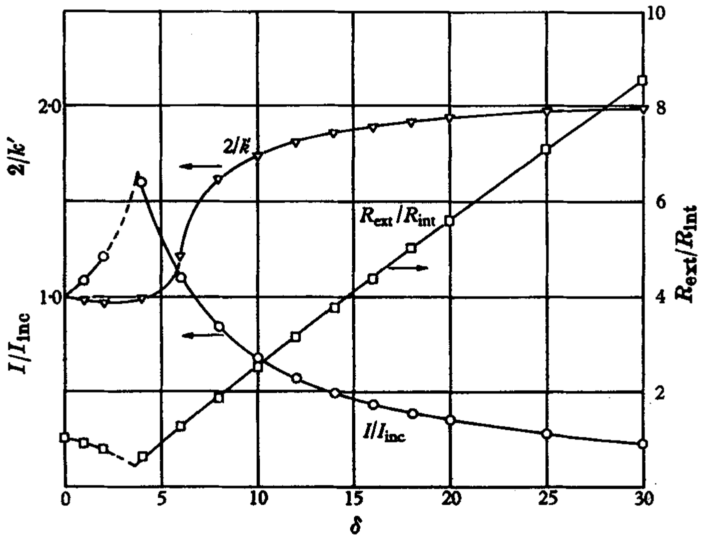

FIGURE 2. Resistance, terminal voltage and current ratios against magnetohydrodynamic interaction parameter δ. $M_0 = 0{\cdot}3$, $\gamma = \frac{5}{3}$.

to some fluid mechanical interaction), that one cannot approximate the interaction by simply treating the fluid as incompressible. For example, it is seen from the figure, that for δ very large, the external load required at optimum power is close to an order of magnitude larger than the corresponding incompressible load.

Figure 3 is a plot of the efficiency of the plasma generator as defined above. As is expected, for small δ, when the effect of the magnetic field on the fluid flow is small, the efficiency is small. It increases as δ increases approaching a value slightly over 20 % in the neighbourhood of δ_{max}.

Finally, let us consider briefly the effect of the interaction on the fluid dynamical variables. The distribution of all of the fluid dynamical variables with distance along the channel was obtained for every δ in each of the two branches with the view to determining whether any of the solutions belonging to the two branches corresponded to any peculiar phenomena, e.g. choking. In every case, the dynamical variables were single valued and varied continuously with channel distance so that behaviour peculiar to choking (Resler & Sears 1958) was absent.

Figure 4 shows the variation of the normalized fluid dynamical variables with fractional distance along the channel for the case $\delta = 20$. It is seen that the pressure and density decrease while the velocity and Mach number increases downstream of the entrance. It appears that, as far as the direction of change of

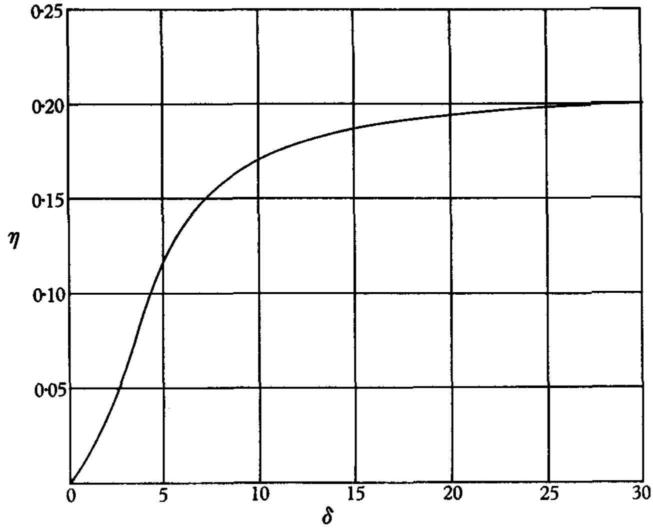

FIGURE 3. Efficiency against magnetohydrodynamic interaction parameter δ. $M_0 = 0{\cdot}3$, $\gamma = \frac{5}{3}$.

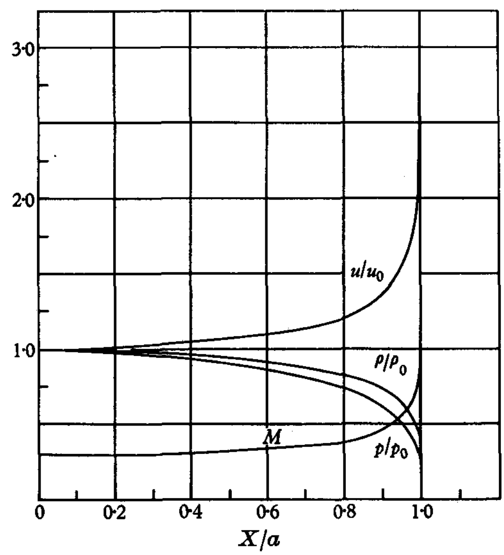

FIGURE 4. Variation of normalized fluid dynamical state variables with dimensionless distance along channel. $M = 0{\cdot}3$, $\gamma = \frac{5}{3}$, $\delta = 20$.

the fluid variables are concerned, the effect of the interaction is analogous both to channel flow with friction and no heat transfer or to channel flow with heat addition and no frictional forces (see, for example, Shapiro 1953). The velocity increase along the channel leads to a very interesting conclusion concerning power delivery to the external load. It was seen in § 3 that the contribution to the power delivered to the external load per unit channel length at station x is $k\sigma\{uB - k/y_0\}$. Since k, σ and B are constants, the power contributed depends on the magnitude of the velocity at that station. We conclude, therefore, that most of the power delivered to the external load comes from the induction taking place near the exit region of the channel.

6. Conclusion

In this paper, solutions for the compressible generator were obtained for only one particular value of the entrance Mach number. It is planned to obtain a family of solutions over a wide range of Mach numbers both subsonic and supersonic. Interest here is not only toward the determination of the effect on the interaction of the various states of compressibility of the entering plasma, but to see whether the asymptotic behaviour occurs for other Mach numbers as well, and at what values of δ. The reason for, and the physical significance, if any, of the asymptotic behaviour itself should be investigated. Further, a thorough examination should be made into the validity of the assumptions made in § 1 directed towards estimating the practicality of the results. In particular, the effect of the possible development of space charge and the formation of space charge sheaths near the walls should be thoroughly investigated.

In conclusion, it is hoped that, while this paper concerned itself mainly with the theoretical treatment of a very highly idealized but interesting problem in hydromagnetics, it will also serve in giving a first insight into the behaviour of a type of generator which may prove to have future practicality.

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